Unified Field Theory From Enlarged Transformation Group. The Covariant Derivative for Conservative Coordinate Transformations and Local Frame Transformations

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Abstract Pandres has developed a theory in which the geometrical structure of a real four-dimensional space-time is expressed by a real orthonormal tetrad, and the group of diffeomorphisms is replaced by a larger group called the conservation group. This paper extends the geometrical foundation for Pandres' theory by developing an appropriate covariant derivative which is covariant under all local Lorentz (frame) transformations, including complex Lorentz transformations, as well as conservative transformations. After defining this extended covariant derivative, an appropriate Lagrangian and its resulting field equations are derived. As in Pandres' theory, these field equations result in a stress-energy tensor that has terms which may automatically represent the electroweak field. Finally, the theory is extended to include 2-spinors and 4-spinors.

1 Introduction

Previously a theory has been presented which exhibits many of the features required for a unified field theory [5, 8, 9]. The main feature is invariance under a group of transformations that is larger than the diffeomorphism group. We will consider a 4-dimensional space X^4 which have local coordinates x^μ ($\mu = 0, 1, 2, 3$) and regard the tetrad h^i_μ (with $i = 0, 1, 2, 3$) as the contracted product of the field variables h^I_μ and L^i_I (defined below). Under the enlarged group of transformations which is defined below, quantities such as the tetrad may be path-dependent. The values of x^i are considered to be inertial coordinates with metric $\eta_{ij} \equiv \text{diag}\{-1, 1, 1, 1\}$ (we use the Einstein summation convention throughout this paper), and the metric tensor is defined by $g_{\mu\nu} = \eta_{ij} h^i_\mu h^j_\nu$. When h^i_μ is a function of x^μ, i.e.

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path-independent, we may interpret X^4 as a 4-dimensional (pseudo-)Riemannian manifold $\mathcal{M}^4$ with metric $g_{\mu\nu}$. This is called the *manifold interpretation*.

A Riemannian manifold is invariant under diffeomorphisms which for $x^\mu \to x^{\bar{\mu}}$ satisfy the property $x^{\bar{\alpha}}_{,\nu,\mu} - x^{\bar{\alpha}}_{,\mu,\nu} = 0$. In the tetrad formulation, it is also invariant under Lorentz transformations $L^{\bar{i}}_i$ which satisfy the condition $\eta_{\bar{i}\bar{j}} = \eta_{ij} L^i_{\bar{i}} L^j_{\bar{j}} = \mathrm{diag}\{-1, 1, 1, 1\}$. The inverse of $L^{\bar{i}}_i$ will be denoted by $L^{\bar{i}}_i$ and hence $L^j_{\bar{i}} L^{\bar{i}}_k = \delta^j_k$ and $L^{\bar{k}}_i L^i_{\bar{j}} = \delta^{\bar{k}}_{\bar{j}}$. Also h^μ_i is defined by the requirement that at every point, $h^\mu_i h^i_\nu = \delta^\mu_\nu$. Under diffeomorphisms on x^μ and Lorentz transformations on x^i, the Riemannian manifold generated by $h^{\bar{i}}_{\bar{\mu}} = h^i_\mu L^{\bar{i}}_i x^\mu_{,\bar{\mu}}$ is the same as that generated by h^i_μ.

When the frame transformation, $L^{\bar{i}}_j$ from one Latin system to another is allowed to be a function of position (local), it is well-known that the transformation from $x^i \to x^{\bar{i}}$ is not a diffeomorphism, i.e. the integrability condition $L^{\bar{i}}_{j,k} - L^{\bar{i}}_{k,j} = 0$ is not satisfied. The value of $x^{\bar{i}}$ will depend on the path in x^i space and hence we cannot interpret the $x^{\bar{i}}$ space as a manifold. The special relativistic equation of a free particle is $\frac{d^2 x^i}{ds^2} = 0$. Under local, non-diffeomorphic, Lorentz transformations $L^{\bar{i}}_i$, this implies that $\frac{d^2 x^{\bar{i}}}{ds^2} = -L^{\bar{i}}_i L^i_{i,j} \frac{dx^i}{ds} \frac{dx^{\bar{j}}}{ds}$ and thus we see that the $x^{\bar{i}}$ system is non-inertial.

Therefore we have three spaces and convert between them using the field variables h^I_μ and L^i_I, with $h^i_\mu = L^i_I h^I_\mu$. Let V_i be a vector in the inertial space,

$$V_i \overset{L^i_I}{\longrightarrow} V_I \overset{h^I_\mu}{\longrightarrow} V_\mu$$

We call the x^i space the inertial space, the x^I space the *internal space* and the x^μ the *world space*. Analogous to the tetrad, we view L^i_I as 4 internal vectors with inverse L^I_i which satisfies $L^I_i L^i_J = \delta^I_J$ and $L^I_j L^j_I = \delta^i_j$. The fundamental fields are L^i_I and h^I_μ since h^i_μ is expressed by $h^i_\mu = L^i_I h^I_\mu$. We will use capital Latin indices such as V^I, h^I_μ, etc. to denote the quantity in the internal system. Note that generally, $L^i_{I,J} - L^i_{J,I} \neq 0$. We require that $\eta_{IJ} = \eta_{ij} L^i_I L^j_J = \mathrm{diag}(-1, 1, 1, 1)$. On the x^I (internal) space, we allow local (nonconstant) Lorentz transformations $L^{\hat{I}}_J$ while on the x^i (inertial) space we allow only global (constant) Lorentz transformations, i.e. $L^{\hat{i}}_{j,\mu} \equiv 0$. We will use the convention that when L has a capital subscript and a lowercase superscript or vice versa, that the L represents the field variable in the given system. When both superscript and subscript are lowercase letters, L will represent a global Lorentz (frame) transformation and when both superscript and subscript are capital letters, L will generally represent a local Lorentz (frame) transformation. When coordinates in the internal space x^I are changed $x^I \to x^{\hat{I}}$, then, in the new system, $h^{\hat{I}}_\mu = h^I_\mu L^{\hat{I}}_I$ and $L^i_{\hat{I}} = L^i_I L^I_{\hat{I}}$. Effectively, the inertial space serves as a pregeometry upon which the richer geometry of the internal space is founded and thence to the external (world) space geometry.

Since $h^I_\mu = h^i_\mu L^I_i$, then $g_{\mu\nu} = \eta_{IJ} h^I_\mu h^J_\nu = \eta_{IJ} h^i_\mu L^I_i h^j_\nu L^J_j = \eta_{ij} h^i_\mu h^j_\nu$. Because the metric is unchanged, the field variables L^i_I do not affect the geometry of the manifold that is determined by h^i_μ. If $h^i_{\mu,\nu} - h^i_{\nu,\mu} = 0$, then, in the manifold interpretation, X^4 is a manifold with a vanishing curvature tensor, but this does not imply that the internal space is flat, since $L^i_{I,J} - L^i_{J,I}$ may be nonzero. This may provide a framework for understanding the geometry of the vacuum.

For transformations on X^4, we consider a larger group of transformations which is called the conservation group [8]. We say a transformation is *conservative* if it satisfies the weaker

condition

$$x^{v}_{,\tilde{\alpha}}\left(x^{\tilde{\alpha}}_{,\mu,v} - x^{\tilde{\alpha}}_{,v,\mu}\right) = 0. \tag{1}$$

The group of conservative transformations contains the diffeomorphisms as a proper subgroup. In the Riemannian manifold interpretation we regard $x^{\bar{\mu}}$ as anholonomic when $x^{\tilde{\alpha}}_{,\mu}$ is non-diffeomorphic. We will use a semicolon to denote covariant differentiation with Christoffel symbol $\Gamma^{\alpha}_{\mu v} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\mu,v} + g_{\sigma v,\mu} - g_{\mu v,\sigma})$. Let $\tilde{V}^{\alpha}$ be a vector density of weight $+1$. The conservation group of transformations arises out of the requirement that a conservation law of the form $\tilde{V}^{\alpha}_{;\alpha} = 0$ is preserved, i.e. $x^{\alpha} \to x^{\tilde{\alpha}}$ being conservative implies that $\tilde{V}^{\tilde{\alpha}}_{;\tilde{\alpha}} = 0$ as well. Dirac [4] has remarked that "further progress lies in the direction of making our equations invariant under wider and still wider transformations." We suggest that this enlargement of the transformation group results in a theory which unifies gravity with the other forces.

As noted above, the field variables L^{I}_{i} generally do not satisfy the integrability condition: $L^{I}_{i,j} - L^{I}_{j,i} = 0$. We define *conservative Lorentz transformations* by the requirement that

$$L^{I}_{\bar{I}}\left(L^{\bar{I}}_{J,I} - L^{\bar{I}}_{I,J}\right) = 0. \tag{2}$$

Since $L^{\bar{I}}_{I}$ is a Lorentz transformation the determinant of $L^{\bar{I}}_{I}$ is ± 1 and hence the derivative of the determinant is zero. This implies that $L^{I}_{\bar{I}}L^{\bar{I}}_{I,J} = 0$ and thus conservative Lorentz transformations satisfy the condition $L^{I}_{\bar{I}}L^{\bar{I}}_{I,I} = 0$. Thus, with use of the chain rule, we have $L^{\bar{I}}_{J}$ conservative $\iff L^{\bar{I}}_{J,\bar{I}} = 0$. However, when we extend the group to complex Lorentz transformations (2) must be used since the determinant of $L^{\bar{I}}_{I}$ is of the form $e^{i\theta(x)}$ and hence is not constant. Although the only diffeomorphic Lorentz transformations are global, there exist local (position-dependent) conservative Lorentz transformations. (The results of this paper do not depend on the concept of conservative Lorentz transformations on x^{I} space, but are included here for future reference.)

We also recall that the Ricci rotation coefficient given by $\gamma^{\alpha}_{\mu v} = h^{\alpha}_{i}h^{i}_{\mu;v}$ is used to define the spin connection. However $\gamma^{\alpha}_{\mu v}$ is not a scalar under local Lorentz transformations $L^{i}_{\bar{i}}$ since

$$h^{\alpha}_{i}h^{i}_{\mu;v} = h^{\alpha}_{i}\left(L^{i}_{\bar{i}}h^{\bar{i}}_{\mu}\right)_{;v}$$

$$= h^{\alpha}_{i}L^{i}_{\bar{i}}h^{\bar{i}}_{\mu;v} + h^{\alpha}_{i}h^{\bar{i}}_{\mu}L^{i}_{\bar{i},v}$$

$$= h^{\alpha}_{\bar{i}}h^{\bar{i}}_{\mu;v} + h^{\alpha}_{i}h^{\bar{i}}_{\mu}L^{i}_{\bar{i},v}.$$

In the manifold interpretation we see that the usual definition of $\gamma^{\alpha}_{\mu v}$ results in a quantity that is not invariant under local frame transformations.

Definition When $h^{i}_{\mu} = L^{i}_{I}h^{I}_{\mu}$ is the tetrad used to define a Riemannian manifold $\mathcal{M}$, we define the extended Ricci rotation coefficient

$$\Upsilon^{\alpha}_{\mu v} \equiv h^{\alpha}_{I}h^{I}_{\mu;v} + h^{\alpha}_{i}h^{I}_{\mu}L^{i}_{I,v}. \tag{3}$$

When L^{i}_{I} is constant, then the second term is zero and we have the usual definition, and also, in this case, we have $\Upsilon^{\alpha}_{\mu v} = h^{\alpha}_{i}h^{i}_{\mu;v}$. Henceforth we will use the symbol $\Upsilon^{\alpha}_{\mu v}$ to mean

the extended Ricci rotation coefficient. One may easily verify that $\Upsilon^{\alpha}_{\mu\nu}$ is a tensor under diffeomorphisms and is a Lorentz scalar. We also have from this definition

$$\Upsilon^{I}_{\mu\nu} = L^{I}_{i}\Upsilon^{i}_{\mu\nu} = L^{I}_{i}h^{i}_{\mu;\nu} = h^{I}_{\mu;\nu} + L^{I}_{i}h^{J}_{\mu}L^{i}_{J,\nu}.$$

2 The Stroke Covariant Derivative

We now define a derivative which is covariant under more general coordinate transformations on x^{μ} as well as local frame transformations on x^{I}. We will call this extended covariant derivative the *stroke covariant derivative* will denote it by use of a vertical stroke. An extended covariant derivative is a standard device used in Gauge theory and in the standard model [10]. We anticipate that our extended covariant derivative will be used to unify gravity with the other forces. When acting on a contravariant vector, the stroke derivative is defined by

$$\begin{aligned} V^{\mu}_{|\nu} &\equiv V^{\mu}_{,\nu} + V^{\beta}h^{\mu}_{i}h^{i}_{\beta,\nu} \\ &\equiv V^{\mu}_{,\nu} + V^{\beta}\left(h^{\mu}_{I}h^{I}_{\beta,\nu} + h^{I}_{\beta}h^{\mu}_{i}L^{i}_{I,\nu}\right). \end{aligned} \tag{4}$$

As stated above, x^{i} is inertial, x^{I} is internal, and the field variables are L^{i}_{I} and h^{I}_{α}. The covariant derivative of the tetrad is $h^{i}_{\mu;\nu} = h^{i}_{\mu,\nu} - h^{i}_{\beta}\Gamma^{\beta}_{\mu\nu}$. Thus $h^{i}_{\mu,\nu} = h^{i}_{\mu;\nu} + h^{i}_{\beta}\Gamma^{\beta}_{\mu\nu}$ and hence $h^{\mu}_{k}h^{k}_{\beta,\nu} = h^{\mu}_{k}h^{k}_{\beta;\nu} + \Gamma^{\mu}_{\beta\nu}$. Thus we have $h^{\mu}_{I}h^{I}_{\beta,\nu} + h^{I}_{\beta}h^{\mu}_{j}L^{j}_{I,\nu} = \Gamma^{\mu}_{\beta\nu} + \Upsilon^{\mu}_{\beta\nu}$, and so the stroke derivative may be written

$$V^{\mu}_{|\nu} = V^{\mu}_{;\nu} + V^{\beta}\Upsilon^{\mu}_{\beta\nu} \tag{5}$$

where $\Upsilon^{\mu}_{\beta\nu}$ is the extended Ricci rotation coefficient defined in (3).

Many investigators have used an alternative covariant derivative with connection given by $L^{\alpha}_{\mu\nu} = h^{\alpha}_{i}h^{i}_{\mu,\nu}$ which is covariant under all coordinate transformations $x^{\mu} \to x^{\bar{\mu}}$, but does not extend to local Lorentz transformations. In [11], the connection for V^{i} is γ^{i}_{jk} which is not equal to our $L^{I}_{j}L^{j}_{J,K}$. Kibble [6] introduces 24 fields A^{ij}_{k} with $A^{ij}_{\mu} = -A^{ji}_{\mu}$ through which a connection $\Gamma^{\alpha}_{\mu\nu}$ is defined. This connection is non-symmetric in its lower indices. Hehl et al. [7] use a connection given by $\Gamma^{k}_{ij} = \{^{k}_{ij}\} - K^{k}_{ij}$, where K^{k}_{ij}, the non-Riemannian part of the connection, is called the contortion. Also, these authors do not use the same Lagrangian as in our theory (usually they use $\int R\sqrt{-g}\,d^{4}x$). Our connection is formed directly from the tetrad h^{I}_{μ} and L^{i}_{I} which are considered to be the fundamental fields. Because of the extended Ricci rotation coefficient, the stroke covariant derivative defined by (4) and (5) is covariant with respect to a wider group of transformations than these other extended covariant derivatives.

For covariant vectors one gets

$$\begin{aligned} V_{\mu|\nu} &= V_{\mu,\nu} - V_{\beta}h^{\beta}_{I}h^{I}_{\mu,\nu} - V_{i}h^{I}_{\mu}L^{i}_{I,\nu} \\ &= V_{\mu;\nu} - V_{\beta}\Upsilon^{\beta}_{\mu\nu} \end{aligned} \tag{6}$$

where, again, x^{i} is assumed to be inertial and the extended Ricci rotation coefficient is used in the second line. Using (5) and (6), one may verify the product rule holds: $(U^{\mu}V_{\nu})_{|\alpha} =$

$U^\mu_{\ |\alpha} V_\nu + U^\mu V_{\nu|\alpha}$. It is also easy to see that $(U^\mu V_\mu)_{|\alpha} = (U^\mu V_\mu)_{,\alpha}$ as would be expected. Analogous formulas hold for tensors of higher rank. For example,

$$\begin{aligned}
V^\alpha_{\ \beta|\mu} &= V^\alpha_{\ \beta,\mu} + V^\gamma_{\ \beta} h^{\ \alpha}_I h^I_{\ \gamma,\mu} + V^I_{\ \beta} h^{\ \alpha}_j L^j_{I,\mu} - V^\alpha_{\ \gamma} h^\gamma_I h^I_{\ \beta,\mu} - V^\alpha_{\ j} h^I_{\ \beta} L^j_{I,\mu} \\
&= V^\alpha_{\ \beta;\mu} + V^\gamma_{\ \beta} \Upsilon^\alpha_{\ \gamma\mu} - V^\alpha_{\ \gamma} \Upsilon^\gamma_{\ \beta\mu}.
\end{aligned} \tag{7}$$

We use (4) to define

$$V^I_{\ |\nu} \equiv h^I_{\ \mu} V^\mu_{\ |\nu} = V^I_{\ ,\nu} + V^J L^I_j L^j_{J,\nu}, \tag{8}$$

and using (6) we have

$$V_{I|\nu} \equiv h^{\ \mu}_I V_{\mu|\nu} = V_{I,\nu} - V_J L^J_j L^j_{I,\nu}. \tag{9}$$

Using the formulas (4)–(9), one may take stroke covariant derivatives of quantities which involve both Latin and Greek indices. Thus

$$V^I_{\ \alpha|\beta} = V^I_{\ \alpha,\beta} + V^K_{\ \alpha} L^I_j L^j_{K,\beta} - V^I_{\ \gamma} h^\gamma_K h^K_{\ \alpha,\beta} - V^I_{\ k} h^K_{\ \alpha} L^k_{K,\beta}.$$

If we apply this result to the field variable $h^I_{\ \alpha}$, noting that $h^I_{\ \mu} h^{\ \mu}_k = L^I_k$, the result is

$$\begin{aligned}
h^I_{\ \alpha|\beta} &= h^I_{\ \alpha,\beta} + h^K_{\ \alpha} L^I_j L^j_{K,\beta} - h^I_{\ \gamma} h^\gamma_K h^K_{\ \alpha,\beta} - L^I_k h^K_{\ \alpha} L^k_{K,\beta} \\
&= h^I_{\ \alpha,\beta} + h^K_{\ \alpha} L^I_j L^j_{K,\beta} - h^I_{\ \alpha,\beta} - L^I_k h^K_{\ \alpha} L^k_{K,\beta} \\
&= 0.
\end{aligned} \tag{10}$$

It is an easy matter to verify that under general coordinate transformations, $V^{\bar\alpha}_{\ |\nu} = x^{\bar\alpha}_{\ ,\mu} V^\mu_{\ |\nu}$ and also under general Lorentz transformations that $V^{\hat I}_{\ |\alpha} = L^{\hat I}_J V^J_{\ |\alpha}$. Hence the stroke derivative of a vector or tensor is another vector or tensor with a rank increased by one.

We also define $V^i_{\ |\nu} \equiv L^i_I V^I_{\ |\nu}$ and $V_{i|\nu} \equiv L^I_i V_{I|\nu}$. These definitions lead to

$$V^i_{\ |\nu} = V^i_{\ ,\nu} \quad \text{and} \quad V_{i|\nu} = V_{i,\nu}, \tag{11}$$

and we easily see that

$$L^i_{I|\nu} = 0.$$

As a check on the consistency of the stroke covariant derivative and the fact that the tetrad is stroke covariant constant we consider whether $\eta_{MN\,|\nu}$ is zero by direct calculation. From (6) with use of the product rule, we have

$$\eta_{MN\,|\nu} = \eta_{MN,\nu} + \eta_{KN} L^j_M L^K_{j,\nu} + \eta_{MK} L^j_N L^K_{j,\nu}.$$

Now $\eta_{MN,\nu} = 0$. Using $L^J_j = \eta^{JK} \eta_{jk} L^k_K$, we see that the second term reduces to the negative of the third term:

$$\begin{aligned}
\eta_{KN} L^j_M L^K_{j,\nu} &= -\eta_{KN} L^j_{M,\nu} L^K_j = -\eta_{KN} \left(\eta^{ij} \eta_{MI} L^I_i \right)_{,\nu} \eta^{KL} \eta_{jk} L^k_L \\
&= -\eta_{KN} \eta^{ij} \eta_{MI} \eta^{KL} \eta_{jk} L^k_L L^I_{i,\nu} \\
&= -\delta^L_N \delta^i_k \eta_{MI} L^k_L L^I_{j,\nu} \\
&= -\eta_{MI} L^i_N L^I_{i,\nu}
\end{aligned}$$

and hence

$$\eta_{MN\,|\nu} = 0.$$

Let $\tilde{V}^\alpha$ be a vector density of weight $+1$ which may be constructed by multiplying a vector V^α by $h = \sqrt{-g}$, the determinant of $h^i_{\ \mu}$. Since $g_{\mu\nu;\alpha} = 0$, then $h_{;\alpha} = 0$. It is also well known that $\tilde{V}^\alpha_{\ ;\alpha} = \tilde{V}^\alpha_{\ ,\alpha}$. Also $h^i_{\ \mu|\nu} = 0$ implies that $\tilde{V}^\alpha_{\ |\alpha} = (h\,V)^\alpha_{\ |\alpha} = h\,V^\alpha_{\ |\alpha}$, and hence one may obtain the following rule for the stroke covariant divergence of vector density of weight $+1$:

$$\tilde{V}^\alpha_{\ |\alpha} = \tilde{V}^\alpha_{\ ,\alpha} + \tilde{V}^\beta \Upsilon^\alpha_{\ \beta\alpha}. \tag{12}$$

Definition The curvature vector (see [8, 9]) is given by

$$C_\mu \equiv \Upsilon^\alpha_{\ \mu\alpha} \tag{13a}$$

The derivative of h is given by $h_{,\alpha} = h\,h^\beta_k\,h^k_{\ \beta,\alpha}$. Since the extended Ricci rotation coefficient is used, this is an extension of Pandres definition, but as its value is the same in the inertial coordinates, x^i, no confusion will arise by using the same symbol, C_μ. Using this and the properties of covariant derivatives and the extended Ricci rotation coefficient one finds that

$$
\begin{aligned}
C_\mu &= h^\alpha_I\,h^I_{\ \mu;\alpha} + h^I_{\ \mu} L^i_{I,i} \\
&= h^\alpha_i \big(h^i_{\ \mu,\alpha} - h^i_{\ \alpha,\mu}\big) \\
&= h^\alpha_I \big(h^I_{\ \mu,\alpha} - h^I_{\ \alpha,\mu}\big) + h^I_{\ \mu} L^J_j \big(L^j_{I,J} - L^j_{J,I}\big)
\end{aligned} \tag{13b}
$$

and

$$
\begin{aligned}
C_i &= -h^{-1}\big(h\,h^\alpha_i\big)_{,\alpha}, \\
C_I &= -h^{-1}\big(h\,h^\alpha_I\big)_{,\alpha} + L^i_{I,i}, \\
C_I &= -H^{-1}\big(H\,H^\alpha_I\big)_{,\alpha} + \Lambda\big(\Lambda^{-1}\Lambda^i_I\big)_{,i}
\end{aligned} \tag{13c}
$$

where the last line, listed here for easy reference, will be explained in the next section. It is easy to verify that C_I transforms as a vector under *all differentiable Lorentz transformations* on the Latin indices, i.e. $C_{\bar{I}} = L^I_{\bar{I}} C_I$, provided $L^i_{\bar{I}}$ is differentiable. However, for C_α to transform as a vector under changes of coordinates, $x^\alpha \to x^{\bar{\alpha}}$, the transformation must be conservative, i.e.

$$C_{\bar{\alpha}} = x^\alpha_{,\bar{\alpha}} C_\alpha \quad \Longleftrightarrow \quad x^\nu_{,\bar{\alpha}}\big(x^{\bar{\alpha}}_{,\mu,\nu} - x^{\bar{\alpha}}_{,\nu,\mu}\big) = 0.$$

3 Complex Lorentz Transformations. Complexification of the Tetrad

We consider allowing the $h^I_{\ \mu}$ and L^I_i to be complex. We will denote complex $h^I_{\ \mu}$ by $H^I_{\ \mu}$ and complex L^I_i by Λ^I_i. Note that $h^i_{\ \mu}$ remains real and thus $g_{\mu\nu}$ remains real. When the Lorentz group is extended to complex values, we will denote the transformation coefficients by $\Lambda^{\bar{I}}_I$. There are two possible ways of extending (see [2]), one in which $\eta_{\hat{i}\hat{j}} = \eta_{IJ}\Lambda^I_{\hat{i}}\Lambda^J_{\hat{j}}$, but we

extend the Lorentz group via the second possibility, i.e.,

$$\eta_{\hat{i}\hat{j}} = \eta_{IJ} \overline{\Lambda_{\hat{i}}^I} \Lambda_{\hat{j}}^J \tag{14}$$

where $\eta_{\hat{i}\hat{j}} = \eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ and a bar over a quantity indicates its complex conjugate. Since $\eta_{\hat{i}\hat{j}}$ is real then $\eta_{\hat{i}\hat{j}} = \overline{\eta_{\hat{i}\hat{j}}} = \eta_{IJ} \Lambda_{\hat{i}}^I \overline{\Lambda_{\hat{j}}^J}$, and we see that $\overline{\Lambda_{\hat{i}}^I}$ is also a Lorentz transformation.

As before, we denote the inverse of Λ_i^I as Λ_I^i and convert between the x^i system and the x^I system as usual, e.g. $V^I = V^i \Lambda_i^I$ and $V^i = V^I \Lambda_I^i$. We also note that $\overline{\Lambda_i^I}$, the complex conjugate of Λ_i^I is also used to convert between the x^i and x^I system and the inverse is the complex conjugate of Λ_I^i, i.e. $\overline{\Lambda_I^I \Lambda_J^i} = \delta_J^I$ and $\overline{\Lambda_j^I \Lambda_I^i} = \delta_j^i$. Let $V_I \equiv V_i \Lambda_I^i$, and $V^I \equiv V^i \Lambda_i^I$. Then $\overline{V_I} = V_i \overline{\Lambda_I^i}$ and $\overline{V^I} = V^i \overline{\Lambda_i^I}$. Similar rules apply for tensors. For the complex tetrad H_α^I, one finds that $H_\alpha^I = \Lambda_i^I h_\alpha^i$ has inverse $H_I^\alpha = h_i^\alpha \Lambda_I^i$ and that $\overline{H_\alpha^I} = \overline{\Lambda_i^I} h_\alpha^i$ has inverse $\overline{H_I^\alpha} = h_i^\alpha \overline{\Lambda_I^i}$. Note that general complexification leads to the condition that $g_{\nu\mu} = \overline{g_{\mu\nu}}$, but because the x^i and x^μ spaces remain real in our construction, $g_{\mu\nu}$ remains real and hence remains symmetric.

The determinant of H_α^I will be denoted by H. We also define $\Lambda \equiv \det(\Lambda_I^i)$ and thus $h = H\Lambda$. When inversions are excluded, and x^I is real, then $\Lambda = 1$, and thus $h = H$; when the Λ_I^i is non-real, then $\Lambda = e^{i\theta}$ and hence $h = He^{i\theta}$, where generally θ is a function of position $\theta(x)$. These comments explain the last line of equations (13d).

When raising or lowering indices, complex conjugation must be used. One finds that $V^I = \eta^{IJ} \overline{V_J}$ and $V_I = \eta_{IJ} \overline{V^J}$. Thus $V^I V_I = \eta^{IJ} \overline{V_J} V_I = \overline{V^I} V_I$. One also finds that $H_\alpha^I = \eta^{IJ} g_{\alpha\beta} \overline{H_J^\beta}$. The definition for the extended Ricci rotation coefficient is $\Upsilon_{\mu\nu}^\alpha = H_I^\alpha H_{\mu;\nu}^I + H_i^\alpha H_\mu^I \Lambda_{I,\nu}^i$, and the curvature vector, C_I, is given by (13d). These quantities are invariant under local Lorentz transformations and conservative transformations on Greek indices. The stroke derivative is invariant under local complex frame transformations.

There are a couple of reasons for extending the group of transformations to include the complex Lorentz transformations. It is well known [2] that the complex Lorentz group which satisfies (14) contains $SU(3)$ as a proper subgroup and that complex quantities are required for $SU(3)$. The complex Lorentz group, Λ, has 16 parameters. Also, the inclusion of spinors and the spinor connection imply that complex Lorentz transformations should be included.

4 The Field Lagrangian

We know that in general relativity we have the property that for a vector density of weight +1, $\tilde{V}_{;\alpha}^\alpha \equiv \tilde{V}_{,\alpha}^\alpha$. Thus an appropriate measure of the new geometry should be

$$\tilde{V}_{|\alpha}^\alpha - \tilde{V}_{,\alpha}^\alpha = \tilde{V}^\alpha C_\alpha. \tag{15}$$

The line of reasoning that leads to this conclusion is as follows. In flat space with a continuously twice-differentiable vector V^α, we have $V_{,\mu,\nu}^\alpha - V_{,\nu,\mu}^\alpha = 0$. Upon replacing the ordinary derivatives by covariant derivative we use $V_{;\mu;\nu}^\alpha - V_{;\nu;\mu}^\alpha = -V^\beta R_{\beta\mu\nu}^\alpha$ to measure the non-flatness of the corresponding Riemannian geometry. The curvature tensor, $R_{\beta\mu\nu}^\alpha$, transforms as a tensor under diffeomorphisms. In a similar way, a space is conservatively flat with respect to the conservation group when $\tilde{V}_{;\alpha}^\alpha - \tilde{V}_{,\alpha}^\alpha = 0$ and hence, after replacing the covariant derivative with the stroke covariant derivative, the non-flatness of the conservation

geometry is measured by (15). The quantity C_μ transforms as a vector under conservative transformations and C_I transforms as a vector under all differentiable Lorentz transformations. We note that there exists a conservative transformation between x^α and $x^{\hat\alpha}$ such that $g_{\hat\alpha\hat\beta} = \text{diag}(-1, 1, 1, 1)$ if and only if $C_\mu = 0$ [8].

A suitable field Lagrangian will be a scalar which is constructed from C_μ. Thus a suitable field Lagrangian is given by

$$\mathcal{L} = \int C^\alpha C_\alpha h d^4 x \tag{16}$$

where $h = \sqrt{-g}$ is the determinant of the tetrad h^i_α. We also have $\mathcal{L} = \int C^i C_i\, h\, d^4 x$ and $\mathcal{L} = \int C^I C_I\, H\Lambda\, d^4 x$. The Riemann tensor is given by $R^\alpha_{\beta\mu\nu} = h^\alpha_i (h^i_{\beta;\mu;\nu} - h^i_{\beta;\nu;\mu})$. Using (3) one finds that the Riemann tensor, the Ricci tensor and the Ricci scalar are given by

$$R^\alpha_{\beta\mu\nu} = \Upsilon^\alpha_{\beta\mu;\nu} - \Upsilon^\alpha_{\beta\nu;\mu} + \Upsilon^\alpha_{\sigma\nu}\Upsilon^\sigma_{\beta\mu} - \Upsilon^\alpha_{\sigma\mu}\Upsilon^\sigma_{\beta\nu} + h^\alpha_I h^i_\beta (\Lambda^I_{i,\mu,\nu} - \Lambda^I_{i,\nu,\mu}),$$

$$R_{\mu\nu} = C_{\mu;\nu} - \Upsilon^\alpha_{\mu\nu;\alpha} + \Upsilon^\alpha_{\sigma\nu}\Upsilon^\sigma_{\mu\alpha} - \Upsilon^\alpha_{\mu\nu}C_\alpha + h^\alpha_I h^i_\mu (\Lambda^I_{i,\alpha,\nu} - \Lambda^I_{i,\nu\alpha}), \tag{17}$$

$$R = 2C^\alpha_{;\alpha} + C^\alpha C_\alpha - \Upsilon^{\alpha\beta\nu}\Upsilon_{\alpha\nu\beta} + \eta^{ij}h^\nu_j h^\alpha_I (\Lambda^I_{i,\alpha,\nu} - \Lambda^I_{i,\nu,\alpha}).$$

Thus one finds that (see [5])

$$C^\alpha C_\alpha = R + \Upsilon^{\alpha\beta\nu}\Upsilon_{\alpha\nu\beta} - 2C^\alpha_{;\alpha} - \eta^{ij}h^\nu_j h^\alpha_I (\Lambda^I_{i,\alpha,\nu} - \Lambda^I_{i,\nu,\alpha}). \tag{18}$$

The additional terms are suggestive of non-gravitational interactions.

Setting $\delta\mathcal{L} = 0$ leads to field equations. The fields that will be varied are H^I_α and Λ^i_I. The requirement that $\eta_{IJ} = \text{diag}(-1, 1, 1, 1)$ and the requirement that $h^i_\mu = H^I_\mu \Lambda^i_I$ be real will not be imposed at the outset by using Lagrange multipliers. Nevertheless the resulting field equations will have solutions with these properties and hence these constraints do not affect the variational problem. Now,

$$\delta(C^I C_I H\Lambda) = (C^I C_I)\Lambda\delta H + (C^I C_I)H\delta\Lambda + 2H\Lambda C^I \delta C_I.$$

Thus, from the formulas

$$\delta H = (-H H^K_\nu)\delta H^\nu_K$$

and

$$\delta\Lambda = \Lambda\Lambda^J_j \delta\Lambda^j_J$$

and using (13d) we easily find that

$$\delta C_I = -H^{-1} H^K_\nu (H H^\alpha_I)_{,\alpha} \delta H^\nu_K - H^{-1}(\delta(H H^\alpha_I))_{,\alpha} + \Lambda\Lambda^J_j (\Lambda^{-1}\Lambda^i_I)_{,i}\delta\Lambda^j_J$$
$$+ \Lambda(\delta(\Lambda^{-1}\Lambda^i_I))_{,i}.$$

When these results are used and an integration by parts is performed, one obtains

$$\delta(C^I C_I H\Lambda) = -2\Lambda C^I (H H^\alpha_I)_{,\alpha} H^K_\nu \delta H^\nu_K + 2(\Lambda C^I)_{,\alpha}\delta(H H^\alpha_I)$$
$$+ 2H\Lambda^2 C^I (\Lambda^{-1}\Lambda^i_I)_{,i}\Lambda^J_j \delta\Lambda^j_J - 2(H\Lambda^2 C^I H^\alpha_i)_{,\alpha}\delta(\Lambda^{-1}\Lambda^i_I)$$
$$- H\Lambda C^I C_I H^K_\nu \delta H^\nu_K + H\Lambda C^I C_I \Lambda^J_j \delta\Lambda^j_J,$$

where the boundary terms have been discarded since $\delta(H_K^\nu) = 0$ and $\delta(\Lambda_I^i) = 0$ on the boundary. After straightforward use of the product rule and chain rule, one obtains

$$\delta(C^I C_I H \Lambda) = 2 H \Lambda \left(\frac{1}{2} C^I C_I H_\nu^K - C^I \Lambda_{I,i}^i H_\nu^K + \Lambda_j^J \Lambda_{J,\nu}^j C^K + C_{,\nu}^K - C_{,I}^I H_\nu^K \right) \delta H_K^\nu$$

$$+ 2 H \Lambda \left(C^I \Lambda \left(\Lambda^{-1} \Lambda_I^i \right)_{,i} \Lambda_j^J - 2 \Lambda_k^K \Lambda_{K,j}^k C^J + 2 \Lambda_k^K \Lambda_{K,I}^k C^I \Lambda_j^J \right.$$

$$\left. - C_{,j}^J + C_{,I}^I \Lambda_j^J + C_j C^J - \frac{1}{2} C^I C_I \Lambda_j^J \right) \delta \Lambda_J^j. \tag{19}$$

Since $h = H \Lambda$ must be nonzero and since δH_K^ν is arbitrary in the region of integration, $\delta \mathcal{L} = 0$ implies that the expression in the first parenthesis in (19) must be zero. Multiplying this expression by H_L^ν one obtains

$$\frac{1}{2} C^I C_I \delta_L^K - C^I \Lambda_{I,i}^i \delta_L^K + \Lambda_j^J \Lambda_{J,L}^j C^K + C_{,L}^K - C_{,I}^I \delta_L^K = 0. \tag{20}$$

The trace of this equation implies that

$$2 C^I C_I - 4 C^I \Lambda_{I,i}^i + C^I \Lambda_j^J \Lambda_{J,I}^j - 3 C_{,I}^I = 0. \tag{21}$$

Similarly the expression in the second parenthesis of (19) must be zero also. Multiplying this expression by Λ_L^j one finds that

$$\delta_L^J C^I \Lambda_{I,i}^i + \delta_L^J C^I \Lambda_k^K \Lambda_{K,I}^k - 2 \Lambda_k^K \Lambda_{K,L}^k C^J - C_{,L}^J + \delta_L^J C_{,I}^I + C^J C_L - \frac{1}{2} \delta_L^J C^I C_I = 0. \tag{22}$$

The trace of this equation yields

$$C^I C_I - 4 C^I \Lambda_{I,i}^i - 2 C^I \Lambda_j^J \Lambda_{J,I}^j - 3 C_{,I}^I = 0 \tag{23}$$

and hence subtracting (23) from (21) gives $C^I \Lambda_j^J \Lambda_{J,I}^j = -\frac{1}{3} C^I C_I$. Also multiplying (21) by 2 and adding to (23) yields $C^I \Lambda_{I,i}^i = \frac{5}{12} C^I C_I - \frac{3}{4} C_{,I}^I$. After inserting these formulae into (20) and (22), one obtains

$$\frac{1}{12} \delta_L^K C^I C_I - \frac{1}{4} \delta_L^K C_{,I}^I + \Lambda_j^J \Lambda_{J,L}^j C^K + C_{,L}^K = 0$$

and

$$-\frac{5}{12} \delta_L^K C^I C_I + \frac{1}{4} \delta_L^K C_{,I}^I - 2 \Lambda_j^J \Lambda_{J,L}^j C^K - C_{,L}^K + C^K C_L = 0.$$

The sum of these two equations yields

$$C^K C_L - \Lambda_j^J \Lambda_{J,L}^j C^K = \frac{1}{3} \delta_L^K C^I C_I.$$

Now since Λ is the determinant of a complex Lorentz transformation, $\Lambda = e^{i\theta}$ and thus $\Lambda_j^J \Lambda_{J,K}^j = \frac{\Lambda_{,K}}{\Lambda} = i \theta_{,K}$. Thus

$$C^K C_L - i \theta_{,L} C^K = \frac{1}{3} \delta_L^K C^I C_I. \tag{24}$$

Now multiply (24) by C_K and sum over K. Assume that $C^K C_K \neq 0$. Then this implies that $C_L - i\theta_{,L} = \frac{1}{3}C_L$ and hence $C_L = \frac{3}{2}i\theta_{,L}$. Substituting this into (24) leads to $C^K C_K = 0$ which contradicts our assumption. Thus we see that our field equations imply that $C^K C_K = 0$.

From (24), we now see that $C^K(C_L - i\theta_{,L}) = 0$. Now assume that $C^K \neq 0$ and substitute $C_L = i\theta_{,L}$ into (20). Then when $K \neq L$, this implies that $i\theta^{,K}_{,L} = \theta^{,K}\theta_{,L}$. But since θ is real then $\theta^{,K}\theta_{,L} = 0$ when $K \neq L$. Thus at most one of the $\theta_{,L}$ is nonzero, but then $C^K C_K = 0$ would imply that all are zero, contradicting the assumption that $C^K \neq 0$.

Hence the field equations imply that $C_I = 0$ and since $C_\alpha = C_I h^I_\alpha$, we have

$$C_\alpha = 0. \tag{25}$$

There are several examples of solutions to the field equations (25). The first example is given by $h^i_\mu = \delta^i_\mu + \delta^i_0 \delta^2_\mu x^1$, where x^1 is a Greek coordinate value, (see [8]), and results in a Ricci scalar value of $R = \frac{1}{2}$. This is equivalent to the pair:

$$h^I_\mu = \delta^I_\mu + \delta^I_0 \delta^2_\mu x^1 \quad \text{and} \quad L^i_I = \delta^i_I.$$

A second example is given by

$$h_i{}^\mu = \delta^\mu_0 \delta^0_i + \delta^\mu_3 \delta^3_i + (\delta^\mu_1 \delta^1_i + \delta^\mu_2 \delta^2_i)\cos x^3 + (\delta^\mu_2 \delta^1_i - \delta^\mu_1 \delta^2_i)\sin x^3, \tag{26}$$

where x^3 is a Greek coordinate. For (26), $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and hence $R^\alpha_{\beta\mu\nu} = 0$, but $\Upsilon^\alpha_{\mu\nu} \neq 0$. A third example is a spherically symmetric solution of the field equations. Let $f(r)$ be a positive differentiable function of $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then the tetrad given by

$$h^i{}_\mu = \delta^i_0 \delta^0_\mu \sqrt{f(r)} + \frac{1}{\sqrt[4]{f(r)}}(\delta^i_1 \delta^1_\mu + \delta^i_2 \delta^2_\mu + \delta^i_3 \delta^3_\mu) \tag{27}$$

yields $C_\mu = 0$ and hence is a solution of the field equations. The metric, in line element form, is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{\sqrt{f(r)}}dr^2 + \frac{r^2}{\sqrt{f(r)}}d\theta^2 + \frac{r^2 \sin^2\theta}{\sqrt{f(r)}}d\phi^2, \tag{28}$$

and both $R^\alpha_{\beta\mu\nu}$ and $\Upsilon^\alpha_{\mu\nu}$ are nonzero.

Using the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, the field equations (25) and symmetrizing (so that $G_{\mu\nu} = G_{\nu\mu}$) we find that

$$G_{\mu\nu} = -\frac{1}{2}\left(\Upsilon^\alpha_{\mu\nu;\alpha} + \Upsilon^\alpha_{\nu\mu;\alpha}\right) + \frac{1}{2}\left(\Upsilon^\alpha_{\sigma\nu}\Upsilon^\sigma_{\mu\alpha} + \Upsilon^\alpha_{\sigma\mu}\Upsilon^\sigma_{\nu\alpha}\right) + \frac{1}{2}g_{\mu\nu}\Upsilon^{\alpha\beta\sigma}\Upsilon_{\alpha\sigma\beta}$$

$$+ \frac{1}{2}\left(h^\alpha_I h^i_\mu(\Lambda^I_{i,\alpha,\nu} - \Lambda^I_{i,\nu,\alpha}) + h^\alpha_I h^i_\nu(\Lambda^I_{i,\alpha,\mu} - \Lambda^I_{i,\mu,\alpha})\right)$$

$$- \frac{1}{2}g_{\mu\nu}\eta^{ij}h^\sigma_j h^\alpha_I(\Lambda^I_{i,\alpha,\sigma} - \Lambda^I_{i,\sigma,\alpha}). \tag{29}$$

These terms on the right suggest that, when interpreted in Riemannian geometry, this new geometry may automatically produce an appropriate stress energy tensor.

5 Inclusion of Spinors. The Spin Connection

The fundamental constant spin tensors, $\sigma^{i\dot{a}}{}_b$, are given as follows [1, 3].

$$\sigma^{0\dot{a}}{}_b = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \sigma^{1\dot{a}}{}_b = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\sigma^{2\dot{a}}{}_b = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \qquad \sigma^{3\dot{a}}{}_b = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{30}$$

We typically will use Latin indices a through f for spin indices (first index refers to the row and the second index refers to the column), and $\sigma^{i\dot{a}b}$ is defined by $\sigma^{i\dot{a}b} = -\sigma^{i\dot{c}}{}_d \mathcal{E}^{db}$ and also $\sigma^i{}_{\dot{a}b} = -\mathcal{E}_{\dot{a}\dot{c}}\sigma^{i\dot{c}}{}_b$, where the spin metric is given by

$$\mathcal{E}^{ab} = \mathcal{E}_{ab} = \mathcal{E}^{\dot{a}\dot{b}} = \mathcal{E}_{\dot{a}\dot{b}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{31}$$

Note that $\mathcal{E}$ is antisymmetric. When we use matrix multiplication to aid in the computation process, we lower indices via a sum on adjacent indices with the matrix for $\mathcal{E}$ afterward (or sum on adjacent indices with the matrix for $-\mathcal{E}$ before the spinor). Similarly, when we use matrix multiplication in raising indices, we sum on adjacent indices with the matrix for $\mathcal{E}$ before (or sum on adjacent indices with the matrix for $-\mathcal{E}$ afterward). Basically, when raising or lowering spinor indices, the summed indices should be adjacent and the sign is $+$ for $\searrow$ and $-$ for $\nearrow$. Useful relations between the σ^i's are:

$$\sigma^i{}_{\dot{a}b}\sigma_j{}^{\dot{a}b} = -\delta^i_j, \quad \sigma^i{}_{\dot{a}b}\sigma_i{}^{\dot{c}d} = -\delta^{\dot{c}}_{\dot{a}}\delta^d_b \quad \text{and} \quad \sigma^{i\dot{a}}{}_c\sigma^{j\dot{c}}{}_b + \sigma^{j\dot{a}}{}_c\sigma^{i\dot{c}}{}_b = \eta^{ij}\delta^{\dot{a}}_b.$$

When the meaning is clear we will suppress the spinor indices, for example σ^i and $\mathcal{E}$.

Generally, for second rank spinors (with 2×2 matrix representation) such as $M^a{}_c$, we have $\mathcal{E}_{ab}M^a{}_c M^b{}_d = \det(M)\mathcal{E}_{cd}$. Thus, if $A^a{}_c$ has determinant $+1$, then $\mathcal{E}_{ab}A^a{}_c A^b{}_d = \mathcal{E}_{cd}$, i.e. the metric is preserved. We will call these $A^a{}_b$ spin transformations and they are elements of $SL(2, \mathbb{C})$. The real Lorentz group is a 6 parameter group as is $SL(2, \mathbb{C})$. As is usual in the tetrad formalism, the fundamental spin tensors are kept constant by coordinating a spin transformation, $A^a_{\dot{b}} \in SL(2, \mathbb{C})$, with the Lorentz transformation L^i_j. Since L^I_i are field variables, these induce field variables A^A_a. This is because we keep $\sigma^{I\dot{A}}_B$ identical to $\sigma^{i\dot{a}}_b$ by coordinating A^A_a with the field variables L^I_i. As noted above, we only allow constant (global) Lorentz transformations, $L^i_{\dot{j}}$, on the x^i (inertial) space and hence we only allow constant $A^a_{\dot{b}}$ on the corresponding inertial spinor space. On the internal space, x^I and its corresponding spinor space, we allow nonconstant (local) Lorentz transformations and nonconstant (local) spin transformations.

Now there is a 1–1 mapping from vectors V^i to rank 2 spinors $V^{\dot{a}b}$ via (31). Specifically $V^{\dot{a}b} = \sigma_i{}^{\dot{a}b}V^i$ which via the relation $\sigma^i{}_{\dot{a}b}\sigma_j{}^{\dot{a}b} = -\delta^i_j$ implies $V^i = -\sigma^i{}_{\dot{a}b}V^{\dot{a}b}$. Since there is coordination between the field variables L^I_I and the induced variables A^A_a, we also have $V^I = -\sigma^I{}_{\dot{A}B}V^{\dot{A}B}$ and $V^{\dot{A}B} = \sigma_I{}^{\dot{A}B}V^I$. Now, because of the constancy of the σ's, $\sigma^i{}_{\dot{a}b,\nu} = 0$ and $\sigma^I{}_{\dot{A}B,\nu} = 0$. From $\sigma^i{}_{\dot{a}b} = \sigma^I{}_{\dot{A}B}L^i_I A^{\dot{A}}_{\dot{a}}A^B_b$, one finds that $\sigma^I{}_{\dot{A}B}(L^i_I A^{\dot{A}}_{\dot{a}}A^B_b)_{,\nu} = 0$. Thus $\sigma^I{}_{\dot{A}B}L^i_{I,\nu} = -\sigma^I{}_{\dot{C}B}L^i_I A^{\dot{a}}_{\dot{A}}A^{\dot{C}}_{\dot{a},\nu} - \sigma^I{}_{\dot{A}C}L^i_I A^a_A A^C_{a,\nu}$. Substituting this into the equation $V^I_{|\nu} = (-\sigma^I{}_{\dot{A}B}V^{\dot{A}B})_{|\nu} = -\sigma^I{}_{\dot{A}B}V^{\dot{A}B}{}_{,\nu} - \sigma^J{}_{\dot{A}B}V^{\dot{A}B}L^I_j L^j_{J,\nu}$, we arrive at the spin form of the stroke covariant derivative of V^I,

$$V^I_{|\nu} = -\sigma^I{}_{\dot{A}B}\left(V^{\dot{A}B}{}_{,\nu} - V^{\dot{C}B}A^{\dot{a}}_{\dot{C}}A^{\dot{A}}_{\dot{a},\nu} - V^{\dot{A}C}A^a_C A^B_{a,\nu}\right). \tag{32}$$

Let a_μ be an arbitrary real vector. One notices that, as in the usual spinor connection, that we may take the replacement $A_B^a A_{a,\nu}^C \to A_B^a A_{a,\nu}^C + i\delta_B^C a_\mu$ which has no effect on (32). This corresponds to the classical Gauge transformation (see [1]). Thus a consistent definition for the stroke derivative of a spinor is given by

$$\Psi_{|\nu}^A = \Psi_{,\nu}^A - \Psi^B \left(A_B^a A_{a,\nu}^A + i\delta_B^A a_\nu \right) \tag{32a}$$

and

$$\Psi_{|\nu}^{\dot A} = \Psi_{,\nu}^{\dot A} - \Psi^{\dot B} \left(A_{\dot B}^{\dot a} A_{\dot a,\nu}^{\dot A} - i\delta_{\dot B}^{\dot A} a_\nu \right). \tag{32b}$$

These definitions imply that $\sigma_{\dot A B|\nu}^I = 0$.

We now consider the extension under parity from the 2-spinor to the 4-spinor. The indices for a 4-spinor will run from 1 to 4 with indices $(1, 2)$ corresponding to *dotted* 2-spinor indices and indices $(3, 4)$ corresponding to *undotted* 2-spinor indices. Let the $n \times n$ zero matrix be denoted by 0_n. Let the matrices for $\sigma^{i\dot a}{}_b$ be briefly denoted by $\boldsymbol{\sigma}^i$, then (in the chiral form) the Dirac matrices, $\gamma^{ia}{}_b$ are given by

$$\gamma^i \equiv \sqrt{2} \begin{bmatrix} 0_2 & \boldsymbol{\sigma}^i \\ \overline{\boldsymbol{\sigma}^i} & 0_2 \end{bmatrix}, \tag{33}$$

where the $\overline{\boldsymbol{\sigma}^i}$ denotes the complex conjugate (i.e. is $\sigma^{ia}{}_b$). One finds that

$$\gamma_i{}^a{}_c \gamma_j{}^c{}_b + \gamma_j{}^a{}_c \gamma_i{}^c{}_b = 2\eta_{ij}\delta_b^a, \tag{34a}$$

or in matrix notation,

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2\eta_{ij} I_4, \tag{34b}$$

where I_4 represents the 4×4 identity matrix. When the signature of the metric is diag$(-1, 1, 1, 1)$, the Klein-Gordon equation is $(\partial^i \partial_i + m^2)\Psi^b = 0$ and the Dirac equation is given by $(\gamma^{ia}{}_b p_i + m\delta_b^a)\Psi^b = 0$. In inertial coordinates, the Dirac equation is $(i\gamma^{ka}{}_b \partial_k + m\delta_b^a)\Psi^b = 0$ and upon multiplying on the left by the operator $i\gamma^j \partial_j$, one finds that the Dirac equation implies the Klein-Gordon equation.

The metric tensor for 4-dimensional spinors is given by

$$\mathcal{E}_{ab} \equiv \delta_a^1 \delta_b^2 - \delta_a^2 \delta_b^1 + \delta_a^3 \delta_b^4 - \delta_a^4 \delta_b^3 \tag{35}$$

and $\mathcal{E}^{ab} = \mathcal{E}_{ab}$. Using (31) we have the matrix form

$$\mathcal{E}_4 \equiv \begin{bmatrix} \mathcal{E} & 0_2 \\ 0_2 & \mathcal{E} \end{bmatrix}.$$

Suppose that $M^a{}_c$ has either of the following special forms:

$$M^a{}_c = \begin{bmatrix} 0_2 & A_2 \\ B_2 & 0_2 \end{bmatrix} \quad \text{or} \quad M^a{}_c = \begin{bmatrix} A_2 & 0_2 \\ 0_2 & B_2 \end{bmatrix}$$

where A_2 and B_2 are 2×2 matrices with $\det(A) = \det(B)$. Using (35), we see that $\mathcal{E}_{ab} M^a{}_c M^b{}_d = \det(A)\mathcal{E}_{cd}$. Hence we define spinor transformations for 4-spinors by

$$\mathbb{A}_{\hat b}^a \equiv \begin{bmatrix} \overline{A} & 0_2 \\ 0_2 & A \end{bmatrix} \tag{36}$$

where $\overline{A}$ is the complex conjugate of A and both are elements of $SL(2, \mathbb{C})$. We also see that there is a mapping between vectors and second rank 4-spinors given by $V^a_b = \frac{1}{2} V_i \gamma^{i\,a}_{\ b}$ with inverse mapping given by $V^i = \frac{1}{2}\gamma^{i\,b}_{\ a}V^a_b$. As with 2-spinors, there is coordination between Lorentz transformations on the Latin indices and spin transformations so that the γ^i remain constant. Similarly, when the field variables L^i_I are given, we require that $\gamma^{IA}_{\ \ B}$ remains unchanged and hence we see that this induces the values of $\mathbb{A}^A_a$. The correspondence is exactly one-to-two, with $\mathbb{A}^A_a$ determined up to a sign. This implies that $(\gamma^{IA}_{\ \ B}\mathbb{A}^a_A\mathbb{A}^B_b L^i_I)_{,\nu} = 0$ and thus $\gamma^{IA}_{\ \ B}(L^i_I\mathbb{A}^a_A\mathbb{A}^B_b)_{,\nu} = 0$. From this we derive that $\gamma^{JA}_{\ \ B}L^i_i L^I_{J,\nu} = -\gamma^{IC}_{\ \ B}\mathbb{A}^A_a\mathbb{A}^a_{C,\nu} + \gamma^{IA}_{\ \ C}\mathbb{A}^C_a\mathbb{A}^a_{B,\nu}$. Thus

$$V^I_{|\nu} = \left(\frac{1}{2}\gamma^{IA}_{\ \ B}V^B_A\right)_{|\nu} = \frac{1}{2}\gamma^{IA}_{\ \ B}\left(V^B_{A,\nu} - V^B_C\mathbb{A}^C_a\mathbb{A}^a_{A,\nu} + V^C_A\mathbb{A}^B_a\mathbb{A}^a_{C,\nu}\right). \tag{37}$$

We note that, for arbitrary vector a_ν, the replacement $\mathbb{A}^B_a\mathbb{A}^a_{C,\nu} \to \mathbb{A}^B_a\mathbb{A}^a_{C,\nu} + i\delta^B_C a_\nu$ has no effect on (37). Thus we define the stroke derivatives of 4-spinors by

$$\Psi^B_{|\nu} \equiv \Psi^B_{,\nu} + \Psi^C\left(\mathbb{A}^B_a\mathbb{A}^a_{C,\nu} + i\delta^B_C a_\nu\right) = \left(\partial_\nu + ia_\nu\right)\Psi^B + \Psi^C\mathbb{A}^B_a\mathbb{A}^a_{C,\nu} \tag{38}$$

and

$$\Psi_{A|\nu} \equiv \Psi_{A,\nu} - \Psi_C\left(\mathbb{A}^C_a\mathbb{A}^a_{A,\nu} + i\delta^C_A a_\nu\right) = \left(\partial_\nu - ia_\nu\right)\Psi_A - \Psi_C\mathbb{A}^C_a\mathbb{A}^a_{A,\nu}. \tag{39}$$

The definition for the stroke derivative of a 4-spinor implies that $\gamma^{IA}_{\ \ B|\nu} = 0$.

6 Concluding Remarks

We have established invertible transformations which convert between the following types

$$
\begin{array}{ccccc}
 & & V^{AB}\ (spinor) & & \\
 & \nearrow & \updownarrow & \searrow & \\
V^i & \leftrightarrow & V^I & \leftrightarrow & V^\mu \\
 & \searrow & \updownarrow & \nearrow & \\
 & & V^I\ (complex) & &
\end{array}
$$
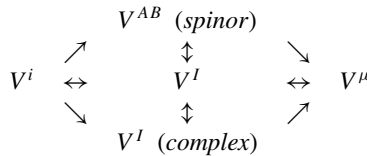
and the stroke covariant derivative of a vector or tensor quantity transforms in the appropriate way.

Let Ψ be a 4-spinor with components Ψ^A. Let D_μ represent the stroke covariant derivative operator. We conjecture that the full Lagrangian is given by

$$
\begin{aligned}
\mathcal{L} &= i\alpha\overline{\Psi}\gamma^\mu D_\mu\Psi + C^\mu C_\mu h \\
&= \alpha i\Psi^\dagger_A(\gamma^0)^A_B(\gamma^\mu)^B_C\Psi^C_{|\mu} + C^\mu C_\mu h
\end{aligned} \tag{40}
$$

where α is an arbitrary real constant and the stroke derivative is given by (38). This Lagrangian is invariant under all conservative coordinate transformations and all differentiable frame transformations. If $\mathbb{A}^a_C$ is constant and if $C_\mu = 0$, the Lagrangian reduces to that of a free particle of spin $1/2$. As the transformations allowed in this new geometry includes local Lorentz transformations, local complex Lorentz transformations, local spin transformations and conservative transformations on Greek indices, we suggest that the geometry has sufficient richness to describe the unification of gravitational, electroweak and strong forces.

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